Hyperbolic Sine

In this problem we study the hyperbolic sine function:

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

reviewing techniques from several parts of the course.

a) Sketch the graph of $y = \sinh x$ by finding its critical points, points of inflection, symmetries, and limits as $x \to \infty$ and $-\infty$.

b) Give a suitable definition for $\sinh^{-1} x$ (the inverse hyperbolic sine) and sketch its graph, indicating the domain of definition.

c) Find $\dfrac{d}{dx} \sinh^{-1} x$.

d) Use your work to evaluate $\displaystyle\int \frac{dx}{\sqrt{a^2 + x^2}}$.

14/9/25

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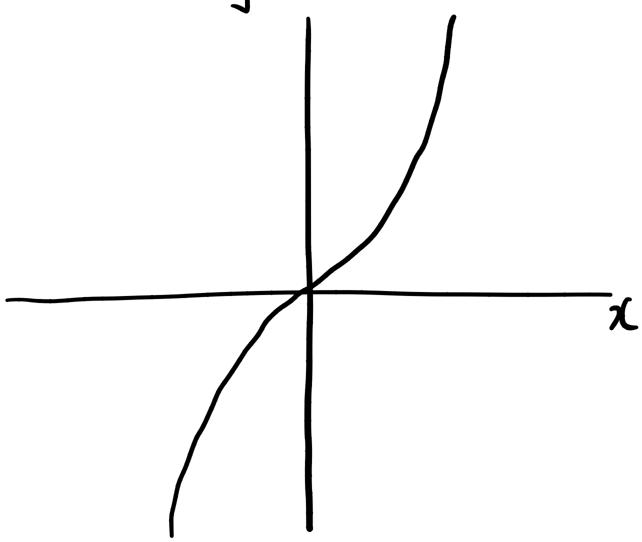
When $x=0$, $y = \frac{1-1}{2} = 0$

a)

$y = \sinh x$



$y = \sinh x$

$y' = \frac{e^x + e^{-x}}{2}$

When $y' = 0$, $\frac{e^x + e^{-x}}{2} = 0$.

$\Rightarrow e^x + e^{-x} = 0$

$e^x = -e^{-x}$

$\Rightarrow u = -\frac{1}{u}$

$u^2 = -1$

$\therefore$ $y = \sinh x$ has no real critical points.

$$\lim_{x\to\infty} \sinh x = \lim_{x\to\infty} \frac{e^x - e^{-x}}{2}$$

$$= \infty$$

$$\lim_{x\to-\infty} \sinh x = -\infty$$

$\sinh(-x)$

$= \frac{e^{-x} - e^x}{2}$

$= -\frac{e^x - e^{-x}}{2}$

$= -\sinh(x)$

$\therefore$ $\sinh(x)$ is an odd function.

b) $y = \sinh x$

$= \frac{e^x - e^{-x}}{2}$

$2y = e^x - e^{-x}$

$2y = u - \frac{1}{u}$

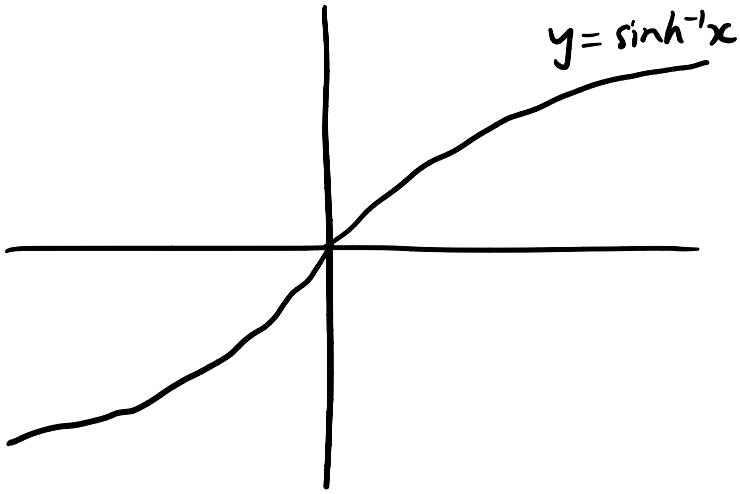
$2yu = u^2 - 1$

$u^2 - 2yu - 1 = 0$

$(u - y)^2 - 1 - y^2 = 0$

$$(u-y)^2 = y^2+1$$
$$u-y = \pm\sqrt{y^2+1}$$
$$u = y \pm \sqrt{y^2+1}$$
$$e^x = y \pm \sqrt{y^2+1}$$
$$x = \ln\left(y \pm \sqrt{y^2+1}\right)$$
$$\therefore\ y = \ln\left(x + \sqrt{x^2+1}\right) = \sinh^{-1}x$$

Domain: $-\infty < x < \infty$



c) $y = \sinh^{-1}x$

$$\sinh y = x$$
$$\Rightarrow \frac{d}{dx}\sinh y = \frac{d}{dx}x$$
$$\cosh y \frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{1}{\cosh y}$$
$$= \frac{1}{\sqrt{1+x^2}}$$

$$\cosh^2 y - \sinh^2 y = 1$$
$$\cosh y = \sqrt{1+\sinh^2 y} \quad (\cosh y \geq 1)$$
$$= \sqrt{1+x^2}$$

d) $$\int \frac{dx}{\sqrt{a^2+x^2}}$$
$$= \int \frac{a\,du}{\sqrt{a^2+a^2u^2}}$$
$$= \int \frac{du}{\sqrt{1+u^2}}$$
$$= \sinh^{-1}u + C$$

Let $x = au$, $dx = a\,du$

$$\therefore \int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\frac{x}{a} + C$$